Homework Assignment On Propositional Logic

Due on November 24, 2020

I leave space between questions to allow you to state your answers consisely.

Exercise 1

For each statement below, say whether it is true or not. You don't need to justify your answer when you say 'true'. When you say 'false', describe a valuation that falsifies the statement.

(1) $p \to (q \to (r \to \neg p)), p \wedge q \vDash \neg r$

(2) $p \to (q \to (r \to \neg q)) \vDash \neg r$

(3) $(p \wedge q) \vee r, p \vDash q$

(4) $p \vee (q \wedge \neg r) \vDash r \to p$

(5) $p \to \neg p \vDash \neg p$

(6) $\neg p \to p \vDash \neg p$

(7) $\vDash (\neg p \to p) \to (\neg p \vee q)$

(8) $\vDash (\neg p \to p) \to (p \vee q)$

Exercise 2

For each of the two formulae below, write a formula that is equivalent to it and whose only connectives are $\neg$ and $\vee$.

(1) $(\neg a \wedge (b \rightarrow c)) \rightarrow \neg(\neg b \vee c)$

(2) $(a \vee \neg(b \wedge c)) \wedge \neg(d \rightarrow (\neg a \wedge e))$

Exercise 3

For each statement below, say whether it is true or false. You don't need to justify your answer when you say 'true'. When you say 'false', give a counterexample (pick two sentences ϕ and ψ which falsify the claim).

Remember that a statement of the form *A if an only if B* means that A cannot be true if B isn't, but also that B cannot be true if A isn't. For instance the arithmetic statement *A number is divisible by 4 if and only if it is divisible by 2* is false, because some numbers are divisible by 2 without being divisible by 4.

(1) A formula of the form $(\phi \vee \psi)$ is satisfiable if and only if ϕ is satisfiable or ψ is satisfiable.

(2) A formula of the form $(\phi \vee \psi)$ is satisfiable if and only if ϕ is satisfiable and ψ is satisfiable.

(3) A formula of the form $(\phi \vee \psi)$ is contradictory if and only if ϕ is contradictory or ψ is contradictory.

(4) A formula of the form $(\phi \vee \psi)$ is contradictory if and only if both ϕ and ψ are contradictory.

(5) A formula of the form $(\phi \wedge \psi)$ is satisfiable if and only if both ϕ and ψ are satisfiable.

(6) A formula of the form $(\phi \wedge \psi)$ is contradictory if and only if ϕ is contradictory or ψ is contradictory.

(7) A formula of the form $(\phi \wedge \psi)$ is contradictory if and only if both ϕ and ψ are contradictory.

(8) A formula of the form $(\phi \wedge \psi)$ is a tautology if and only if ϕ is a tautology or ψ is a tautology.

(9) A formula of the form $(\phi \wedge \psi)$ is a tautology if and only if both ϕ and ψ are tautologies.

Exercise 4

For each set of formulae below, describe all the valuations that satisfy it (no need to justify your answer).

(1) $\{p_1, p_2 \to p_1, p_3 \to (p_2 \to p_1)\}$

(2) $\{\neg p_1, \neg(p_2 \to p_1), \neg(p_3 \to (p_2 \to p_1))\}$

(3) The following infinite set:
$\{\neg p_1, \neg(p_2 \to p_1), \neg(p_3 \to (p_2 \to p_1)), \neg(p_4 \to (p_3 \to (p_2 \to p_1))), \ldots,$
$\neg(p_i \to (p_{i-i} \to (\ldots \to (p_2 \to p_1))) \ldots), \ldots\}$

Exercise 5 - Harder

For this exercise, you will need to use a separate sheet.

We consider a propositional language based on only three atomic sentences p_1, p_2 and p_3, and which has two unary connectives $\neg$ (with its standard semantics) and O (whose semantics is described below), and the standard binary connectives (with their standard semantics) as well as a connective for exclusive disjunction, noted $\veebar$.

To each valuation v we associate the set of atoms that v maps to true, noted $pos(v)$. For instance, if v assigns 1 to p_1 and p_3 and 0 to p_2, then $pos(v) = \{p_1, p_3\}$.

We now define an ordering relation on valuations by:

$u \leq v$ if $pos(u) \subseteq pos(v)$

In words: u is 'smaller' than v if every atom that is true in u is also true in v (but possibly v makes true more atoms than u).

The syntax and semantics of O are given by:

(1) a. If ϕ is a formula, then $O\phi$ is a formula.
    b. For any sentence ϕ and any valuation v, $v(O\phi) = 1$ if:
       (i) $v(\phi) = 1$ and
       (ii) there is no valuation u such that $u(\phi) = 1$ and $u < v$. (where $u < v$ means $u \leq v$ and $u \neq v$)
       And otherwise $v(O\phi) = 0$

(2) In words: $O\phi$ is true relative to valuation v if v makes ϕ true, and there is no 'smaller' valuation u that also makes ϕ true. Otherwise it is false.

1. Find a formula equivalent to Op_2 in which O does not occur.

2. Find a formula equivalent to $O(p_1 \vee p_2)$ in which O does not occur.

3. Describe the set of valuations that make $O(p_1 \vee (p_2 \vee p_3))$ true.

4. Is any of the following formulae equivalent to $O(p_1 \vee (p_2 \vee p_3))$?

   (a) $p_1 \veebar (p_2 \veebar p_3)$

   (b) $p_1 \veebar (p_2 \vee p_3)$

   (c) $p_1 \vee (p_2 \veebar p_3)$

   (d) $p_1 \vee (p_2 \vee p_3)$

5. Describe all the valuations that make $O(p_1 \vee (p_2 \wedge p_3))$ true.

6. Is $O(p_1 \vee (p_2 \wedge p_3))$ equivalent to $(p_1 \veebar (p_2 \wedge p_3))$?

7. Is it possible to characterize the meaning of O by means of a truth-table?